RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 4758

Differential Equations

Friday **27 JANUARY 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
MEI Examination Formulae and Tables (MF2)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is 72.

This question paper consists of 3 printed pages and 1 blank page.

1 In an electric circuit, the current, I amps, at time t seconds is modelled by the differential equation

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + 6\frac{\mathrm{d}I}{\mathrm{d}t} + kI = 6\mathrm{e}^{-t},$$

where k is a positive constant which depends on the capacitor in the circuit.

(i) In the case $k = 8$, find the general solution. [8]

(ii) In the case $k = 9$, find the solution given that initially the current is 1.5 amps and $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$.
State the limiting value of the current as t tends to infinity. [12]

(iii) Show that, for all positive values of k, the complementary function for this differential equation will tend to zero as t tends to infinity. [4]

2 Three differential equations are to be solved.

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 1 \qquad (1)$$

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3xy = \cos x \qquad (2)$$

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3x(y + 0.1y^2) = \cos x \qquad (3)$$

(i) By separating the variables, or otherwise, solve equation (1) to find y in terms of x, subject to the condition $y = 0$ when $x = 1$. Hence calculate y when $x = 2$, giving your answer correct to three significant figures. [8]

(ii) Solve equation (2) to find y in terms of x, subject to the condition $y = 0$ when $x = 1$. Hence calculate y when $x = 2$, giving your answer correct to three significant figures. [11]

Euler's method is used to solve equation (3). The algorithm is given by

$$x_{r+1} = x_r + h,$$
$$y_{r+1} = y_r + hy'_r.$$

The algorithm starts from $x = 1, y = 0$ with $h = 0.1$, and gives $y = 0.034\,411$ when $x = 1.8$.

(iii) Carry out two more steps of the algorithm to find an approximation for the value of y when $x = 2$. How could you find this value with greater accuracy? [5]

3 A rock of mass m kg is dropped from a height of 50 m above the sea. The rock falls under the action of its weight mg N and a resistance force R N, given by $R = 0.001mv^2$, where v m s^{-1} is the velocity of the rock.

At time t seconds, the rock has fallen a distance x m.

(i) Show that Newton's second law gives the equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = g - 0.001v^2,$$

justifying the signs of the terms. [3]

(ii) Solve this differential equation to find v in terms of x. Hence show that the rock hits the water at a speed of 30.54 m s^{-1}, correct to two decimal places. [7]

When the rock is in the water, the resistance to motion is modelled by $R = 2mv$. Assume that there is no instantaneous change in the velocity of the rock as it hits the water, and that the only forces on the rock are its weight and the resistance.

(iii) Formulate and solve a differential equation to find a relationship between v and x while the rock is under water (you are **not** required to find v in terms of x). How deep must the water be in order for the velocity of the rock to be reduced to 5 m s^{-1}? [9]

(iv) Use your differential equation from part **(iii)** to deduce the terminal velocity of the rock under water. Sketch a graph of v against x for the entire motion of the rock. [5]

4 The following simultaneous differential equations are to be solved.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - 2y + \sin t$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 3y + \cos t$$

(i) Show that $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\dfrac{\mathrm{d}x}{\mathrm{d}t} + 5x = 3\sin t - \cos t$. [5]

(ii) Find the general solution for x in terms of t. [10]

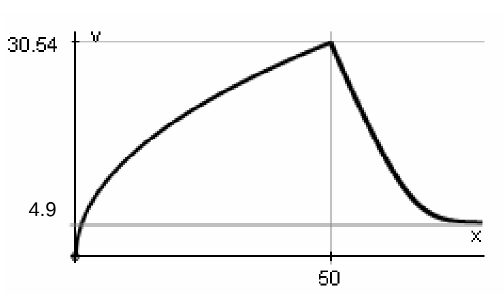
(iii) Hence obtain the corresponding general solution for y. [5]

(iv) Obtain approximate expressions for x and y in terms of t, valid for large t. Hence show that, for large t, x is approximately equal to y. Show that, for small t, this is not necessarily the case. [4]

Mark Scheme 4758
January 2006

1(i)	$\lambda^2+6\lambda+8=0$	M1		
	$\lambda=-2$ or -4	A1		
	CF $I=A\mathrm{e}^{-2t}+B\mathrm{e}^{-4t}$	F1		
	PI $I=a\mathrm{e}^{-t}$	B1		
	$a\mathrm{e}^{-t}-6a\mathrm{e}^{-t}+8a\mathrm{e}^{-t}=6\mathrm{e}^{-t}$	M1	differentiate and substitute	
	$3a=6$	dM1	compare	
	$a=2$	A1		
	$I=2\mathrm{e}^{-t}+A\mathrm{e}^{-2t}+B\mathrm{e}^{-4t}$	F1	CF + PI	
				8
(ii)	$\lambda^2+6\lambda+9=0$	M1		
	$\lambda=-3$ (repeated)	A1		
	CF $I=(C+Dt)\mathrm{e}^{-3t}$	F1		
	PI $I=b\mathrm{e}^{-t}$	B1		
	$b\mathrm{e}^{-t}-6b\mathrm{e}^{-t}+8b\mathrm{e}^{-t}=6\mathrm{e}^{-t}$	M1	substitute and compare	
	$b=\frac{3}{2}$	A1		
	$I=\frac{3}{2}\mathrm{e}^{-t}+(C+Dt)\mathrm{e}^{-3t}$	F1	CF + PI	
	$1.5=\frac{3}{2}+C\Rightarrow C=0$	M1	condition on I	
	$\dot{I}=-\frac{3}{2}\mathrm{e}^{-t}-3(C+Dt)\mathrm{e}^{-3t}+D\mathrm{e}^{-3t}$	M1	differentiate	
	$0=-\frac{3}{2}-3C+D\Rightarrow D=\frac{3}{2}$	M1	condition on $\dot{I}$	
	$I=\frac{3}{2}\left(\mathrm{e}^{-t}+t\,\mathrm{e}^{-3t}\right)$	A1	cao	
	as $t\to\infty, I\to 0$	F1	recognise $\mathrm{e}^{-t}\to 0$	
				12
(iii)	$\lambda^2+6\lambda+k=0\Rightarrow\lambda=-3\pm\sqrt{9-k}$	M1		
	$0<k<9\Rightarrow\sqrt{9-k}<3\Rightarrow$ two negative roots			
	hence $I=A\mathrm{e}^{-\lambda_1 t}+B\mathrm{e}^{-\lambda_2 t}\to 0$	E1		
	$k>9\Rightarrow\lambda=-3\pm\beta j$	M1	complex roots with negative real part (or CF)	
	$\mathrm{e}^{-3t}(A\cos\beta t+B\sin\beta t)\to 0$	E1		
				4

2(i)	$\frac{1}{1-3y}\frac{dy}{dx}=\frac{1}{x^2}$	M1	separate
	$\int\frac{1}{1-3y}dy=\int\frac{1}{x^2}dx$	M1	integrate
	$-\frac{1}{3}\ln\lvert 1-3y\rvert=-\frac{1}{x}+c$	A1 A1	± LHS ± RHS
	$1-3y=Ae^{3/x}$	M1	rearrange
	$y=0, x=1\Rightarrow A=e^{-3}$	M1	condition
	$y=\frac{1}{3}\left(1-\exp(\frac{3}{x}-3)\right)$	A1	
	$x=2\Rightarrow y\approx 0.259$	A1	from correct solution
			8
(ii)	$\frac{dy}{dx}+\frac{3}{x}y=\frac{1}{x^2}\cos x$	M1	divide
	$I=\exp(\int\frac{3}{x}dx)$	M1	attempt integrating factor
	$=x^3$	A1	
	$\frac{d}{dx}\left(x^3y\right)=x\cos x$	F1	follow their I
	$x^3y=\int x\cos x\,dx=x\sin x-\int\sin x\,dx$	M1	integrate (by parts)
	$=x\sin x+\cos x+B$	A1	RHS (or multiple) constant not required here
	$y=x^{-2}\sin x+x^{-3}(\cos x+B)$	M1	divide to get y
	$x=1, y=0\Rightarrow B=-\cos 1-\sin 1$	M1	use condition
	$y=x^{-2}\sin x+x^{-3}(\cos x-\cos 1-\sin 1)$	A1	
	$x=2\Rightarrow y\approx 0.00258$	M1 A1	substitute $x=2$ cao
			11
(iii)	$y'=\left(\cos x-3x\left(y+0.1y^2\right)\right)x^{-2}$	B1	seen or implied by correct numerical value
	x y $\dot{y}$		
	1.8 0.034411 –0.12767	M1	use algorithm
	1.9 0.021644 –0.12380	A1	$y(1.9)$
	2.0 0.00926(3…)	A1	$y(2.0)$
	Using smaller h would give greater accuracy	B1	
			5

3(i)	$F = ma \Rightarrow mv\dfrac{dv}{dx} = mg - 0.001mv^2$	M1	N2L (accept just *ma* for M1)
	$\Rightarrow v\dfrac{dv}{dx} = g - 0.001v^2$	E1	
	down positive so weight positive and resistance negative as it opposes motion	B1	
			3
(ii)	$\displaystyle\int \frac{-0.002v}{g - 0.001v^2}\,dv = \int -0.002\,dx$	M1 M1	separate integrate
	$\ln\left\lvert g - 0.001v^2 \right\rvert = -0.002x + c$	A1	LHS (or multiple)
	$v^2 = 1000\left(g - Ae^{-0.002x}\right)$	M1	rearrange
	$x = 0, v = 0 \Rightarrow A = g$	M1	use condition
	$v = \sqrt{1000g\left(1 - e^{-0.002x}\right)}$	A1	cao
	$x = 50 \Rightarrow v = 30.54$	E1	must follow correct work
			7
(iii)	$mv\dfrac{dv}{dx} = mg - 2mv$	M1	
	$v\dfrac{dv}{dx} = g - 2v$	A1	
	$\displaystyle\int \frac{v}{g - 2v}\,dv = \int dx$	M1	separate
	$\frac{1}{2}\displaystyle\int\left(-1 + \frac{g}{g - 2v}\right)dv = x + c$	A1 M1	$x + c$ attempt to integrate LHS
	$-\frac{1}{2}v - \frac{1}{4}g\ln\lvert g - 2v\rvert = x + c$	A1	
	$x = 50, v = 30.54 \Rightarrow c = -74.91...$	M1	use condition (correct value of *v* at least)
	$v = 5 \Rightarrow x = 76.36$	M1	
	so 26.4 m deep	A1	
			9
(iv)	terminal velocity when acceleration zero	M1	
	$\Rightarrow v = 4.9$	F1	follow their DE
		B1 B1 B1	increasing from (0,0) decreasing to asymptote at 4.9 (or follow their value) cusp/max at (50, 30.54) (both coordinates shown)
			5

4(i)	$\ddot{x} = \dot{x} - 2\dot{y} + \cos t$	M1	differentiate	
	$= \dot{x} - 2(4x - 3y + \cos t) + \cos t$	M1	substitute	
	$= \dot{x} - 8x + 6y - \cos t$			
	$y = \frac{1}{2}(x + \sin t - \dot{x})$	M1	y in terms of $x, \dot{x}$	
	$\ddot{x} = \dot{x} - 8x + 3(x + \sin t - \dot{x}) - \cos t$	M1	substitute	
	$\ddot{x} + 2\dot{x} + 5x = 3\sin t - \cos t$	E1		
				5
(ii)	$\lambda^2 + 2\lambda + 5 = 0$	M1	auxiliary equation	
	$\lambda = -1 \pm 2j$	M1	solve to get complex roots	
		A1		
	CF $x = e^{-t}\left(A\cos 2t + B\sin 2t\right)$	F1	CF for their roots (for complex roots must be in exp/trig form, not complex exponentials)	
	PI $x = a\sin t + b\cos t$	B1		
	$\dot{x} = a\cos t - b\sin t, \ddot{x} = -a\sin t - b\cos t$	M1	differentiate twice and substitute	
	$-a - 2b + 5a = 3$	M1	compare	
	$-b + 2a + 5b = -1$	M1	solve	
	$a = \frac{1}{2}, b = -\frac{1}{2}$	A1		
	$x = e^{-t}\left(A\cos 2t + B\sin 2t\right) + \frac{1}{2}\left(\sin t - \cos t\right)$	F1	CF + PI	
				10
(iii)	$y = \frac{1}{2}\left(x + \sin t - \dot{x}\right)$	M1	y in terms of $x, \dot{x}$	
	$\dot{x} = -e^{-t}\left(A\cos 2t + B\sin 2t\right) + e^{-t}\left(-2A\sin 2t + 2B\cos\right.$			
	$+\frac{1}{2}\left(\cos t + \sin t\right)$	M1	differentiate x	
	$y = e^{-t}\left((A-B)\cos 2t + (A+B)\sin 2t\right) + \frac{1}{2}\left(\sin t - \cos t\right)$	M1	substitute for $x, \dot{x}$	
		A1	CF part	
		A1	PI part	
				5
(iv)	$x \sim \frac{1}{2}\left(\sin t - \cos t\right)$	F1		
	$y \sim \frac{1}{2}\left(\sin t - \cos t\right)$	F1		
	hence for large $t, x \approx y$	B1	must follow correctly from their solutions	
	but unless $A = B = 0, \quad A - B \neq A$ or $A + B \neq B$ so $x \neq y$	B1	must follow correctly from their solutions	
				4

4758: Differential Equations (Written Examination)

General Comments

The standard of work was generally good. Questions 1 and 4 were attempted by almost all of the candidates. Most then chose question 2 rather than question 3. Candidates often produced accurate work; however errors in integration were common.

Comments on Individual Questions

1) (i) This was often completely correct.

(ii) Many correct solutions were seen, but some candidates could not state the correct complementary function associated with a repeated root of the auxiliary equation. When considering the behaviour as t tends to infinity, it was recognised that candidates may not know the behaviour of the $t\,\mathrm{e}^{-3t}$ term, and so credit was given on the basis of how they dealt with the other term.

(iii) When considering the complementary function in the general case, many candidates omitted to consider complex roots of the auxiliary equation. When considering the real roots, candidates often did not explain why both roots must be negative.

2) (i) This was often done well, except for slips in the integration. However a minority of students ignored the suggestion to separate variables and when using the integrating factor method found the resulting integral difficult.

(ii) The integrating factor method was usually applied and understood, but errors were common, in particular with the integration by parts and either omitting the constant or failing to divide it by the integrating factor when expressing y in terms of x.

(iii) The Euler calculation was often done well but some worked in degrees and others produced unrecognisable figures with no indication of method.

3) (i) Most candidates were able to use Newton's second law to obtain the differential equation, but explanations of the signs were often vague.

(ii) Candidates tried various methods to solve the differential equation, and it was common for separation of variables to be ignored. Even those who used the correct method often made errors in integration.

(iii) This differential equation also caused problems for candidates. Many did not identify the correct method, and even among those who did, correct solutions were extremely rare.

(iv) Many were able to deduce the terminal velocity. The standard of graph sketching was very variable. Some did not consider the entire motion, some ignored the initial conditions, but some produced good sketches. Even some of those who had been unable to solve the differential equations gained full credit here by deducing the key features of the graph from the given information.

4) (i) The elimination of y was often done well, although a few differentiated the first equation with respect to x rather than t.

(ii) The solution was often done well, although minor slips were common.

(iii) Many candidates correctly used the first equation. Some tried to set up and solve a differential equation for y; such attempts never tried to relate the arbitrary constants in the two solutions.

(iv) The limiting expressions were usually well done, but many candidates did not make clear conclusions in either case.